Discrete qualocation methods for logarithmic-kernel integral equations on a piecewise smooth boundary

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Abstract

We consider a fully discrete qualocation method for Symm's integral equation. The method is that of Sloan and Burn [14], for which a complete analysis is available in the case of smooth curves. The convergence for smooth curves can be improved by a subtraction of singularity (Jeon and Kimn [10]). In this paper we extend these results for smooth boundaries to polygonal boundaries. The analysis uses a mesh grading transformation method for Symm's integral equation, as in Elschner and Graham [4] and Elschner and Stephan [7], to overcome the singular behavior of solutions at corners.

1 Introduction

Many methods have been proposed for the logarithmic-kernel integral equation on closed curves, but often with the unrealistic assumption that the curve is smooth. This is the case, for example, for the fully discrete ("discrete qualocation") method of [14], [13] and the modification [10]. Our main aim in this paper is to extend these methods to curves with corners.

We consider the Laplace equation with Dirichlet boundary data on a simply connected domain Ω. We assume the boundary Γ is polygonal and $\mathrm{Cap}(\Gamma) \neq 1$. Let

$$\Gamma = \Gamma_1 \cup \Gamma_2 \cup \cdots \cup \Gamma_I,$$

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where $\{\Gamma_i\}$ are smooth pieces of Γ, and $\{s_0, \cdots, s_I = s_0\}$ are corner points on Γ. Consider the Dirichlet problem

$$\begin{aligned} \Delta\phi &= 0 \text{ in } \Omega \\ \phi &= g \text{ on } \Gamma = \partial\Omega. \end{aligned} \tag{1.1}$$

Representing ϕ as a single layer potential

$$\phi(t) = -\frac{1}{\pi}\int_\Gamma \log|t-s| v(s)\, dl_s, \quad t \in \Omega, \tag{1.2}$$

and using the continuity of the single layer potential up to boundary, we obtain the logarithmic-kernel integral equation

$$g(t) = -\frac{1}{\pi}\int_\Gamma \log|t-s|\, v(s)\, dl_s, \quad t \in \Gamma, \tag{1.3}$$

where we seek the single layer density function v on Γ. It is well-known that v has singularities at corners even with a smooth g [8], [12]. The regularity result states that

$$v(s) = \sum_{i=0}^{I-1} a_i|s-s_i|^{r_i} + \text{ a smoother function}, \quad r_i = \frac{\pi}{\pi + |\pi - \theta_i|} - 1, \tag{1.4}$$

where θ_i is the interior angle at s_i, and $|s - s_i|$ represents the arc length.

As a first step, we introduce a parametrization of the boundary Γ. Let $a(x)$ be a parametrization of Γ such that $a(x_i) = s_i$ for $i = 0, \ldots, I$, where

$$0 = x_0 < x_1 < \cdots < x_I = 1, \tag{1.5}$$

so that the subinterval $[x_{i-1}, x_i]$ corresponds to the boundary segment $\Gamma_i \subset \Gamma$ under a, and $|a'| \neq 0$ for $x \in (x_{i-1}, x_i)$. Now we choose a mesh-grading parameter $q \geq 2$ and a mesh-grading transformation $\gamma : [0,1] \to [0,1]$ such that γ is bijective, and

$$\gamma^{(i)}(0) = \gamma^{(i)}(1) = 0,\ i = 1, \ldots, q-1.$$

For example Kress [11] considered

$$\gamma(x) = \frac{\nu^q(x)}{\nu^q(x) + \nu^q(1-x)}, \tag{1.6}$$

with

$$\nu(x) = \left(\frac{1}{2} - \frac{1}{q}\right)(2x-1)^3 + \frac{1}{q}(2x-1) + \frac{1}{2},\ q \geq 2.$$

Then the parametrization transformed to the subinterval $[x_{i-1}, x_i]$, namely

$$\alpha(x) := a\left(x_{i-1} + (x_i - x_{i-1})\gamma\left(\frac{x - x_{i-1}}{x_i - x_{i-1}}\right)\right),\ x_{i-1} \leq x \leq x_i,\ i = 1, \ldots, I, \tag{1.7}$$

satisfies

$$\alpha^{(j)}(x_i) = 0,\ j = 1, \ldots, q-1,\ i = 1, \ldots, I. \tag{1.8}$$

Substituting $t = \alpha(x)$ and defining $z(x) = v(\alpha(x))\alpha'(x)/(2\pi)$ and $f(x) := g(\alpha(x))$, Equation (1.3) becomes

$$-2\int_0^1 \log|\alpha(x) - \alpha(y)|\, z(y)\, dy = f(x). \tag{1.9}$$

Because of the factor $|\alpha'(x)|$, the new solution z will be smoother; in fact we see that

$$z(x) = \sum_{i=0}^{I-1} c|x - x_i|^{q(1+r_i)-1} + \text{ smoother terms}, \tag{1.10}$$

where r_i is defined in (1.4). For this reason we may hope that z will admit a higher order of convergence of an approximate solution.

An important aspect of the parametrization that we have so far passed over is the choice of $x_0, \ldots, x_{I-1}$, the preimages of the corners. They should be chosen so that

$$x_{i+1} - x_i = |s_{i+1} - s_i|^{1/q} / \sum_{j=0}^{I-1} |s_{j+1} - s_j|^{1/q},$$

so that practice and theory match. This choice ensures that there holds

$$\lim_{x \to x_i} |x - x_i|^{1-q} |ds/dx| > 0.$$

The mesh grading transformation method has been extensively used for second kind integral equations [9], [11]. For first kind integral equations mesh grading transformations have recently been used by Elschner and Graham for the spline collocation [4] and quadrature methods [5]; by Elschner and Stephan [7] for the trigonometric polynomial collocation and discrete collocation methods; and by Elschner, Prössdorf and Sloan [6] for spline qualocation methods. In all of the papers the use of a mesh-grading transformation together with a uniform mesh has allowed the use of Fourier methods to analyse the principal term (i.e., the term which would represent the operator in the case of a circular contour Γ), together with Mellin convolution arguments to handle the difficulties introduced by the corners.

In the present paper we use similiar methods to extend the discrete qualocation methods of [14] and [10], previously analysed only for smooth curves, to curves with corners. In Section 2 the method of Sloan and Burn [14] is reviewed, and in Section 3 a modification of the method due to Jeon and Kimn [10] is presented. Section 4 deals with the stability analysis of the case of a polygonal boundary. In Section 5 an error analysis is presented both for the L_2 norm and for certain linear functionals, for which one additional order of convergence can be proved. Section 6 is devoted to numerical results.

The analysis in this paper follows closely that of Elschner and Stephan in [7] for the dicrete collocation method, but the present analysis goes beyond that in [7] in that it obtains results not only in the L_2 norm, but also (and often with one power of h more) for certain linear functionals. The results for linear functionals hold also for the discrete collocation method.

2 Review of the Discrete Qualocation Method

In this section we review the discrete qualocation method for smooth curves of [14], [13] and [10]. Sloan and Burn [14] proposed the method, and provided an analysis by Fourier series. Saranen and Sloan [13] showed that the results obtained by the Fourier series analysis extend without loss to arbitrary smooth curves. Jeon and Kimn [10] introduced an improved treatment of the logarithmic singularity through an subtraction of the singularity (see Section 3).

We start by introducing some notation. Let $\mathbb{Z}$ be the set of integers, and $\mathbb{Z}^* = \mathbb{Z}\backslash\{0\}$. For given $N \in \mathbb{Z}$ we define $h = 1/N$, and $\Lambda_h = \{\mu \in \mathbb{Z} : -\frac{N}{2} < \mu \leq \frac{N}{2}\}$, and assume that each corner preimage $x_i, i = 0, ..., I-1$, is a multiple of h. Our trial space T_h is a space of trigonometric polynomials, $T_h = \text{span}\{e^{2\pi i\mu x} : \mu \in \Lambda_h, x \in [0,1]\}$, and our test space S_h^r is the space of 1-periodic smoothest splines of order r with uniformly spaced nodes $\{kh : 0 \leq k \leq N-1\}$. In contrast to the trigonometric trial space, the test space S_h^r has a local basis: for example, S_h^1 is the space of piecewise constant functions, S_h^2 is spanned by the hat functions

$$v_k(x) = \begin{cases} 1 - |x - kh|/h, & x \in [(k-1)h, (k+1)h], \\ 0, & \text{otherwise}, \end{cases}$$

and S_h^4 is the space of cubic splines.

Qualocation methods are characterised by special quadrature approximations of the inner-product integral $(f, g) = \int_0^1 f(x)\overline{g(x)}\,dx$. Thus for f, g any 1-periodic continuous functions we define

$$(f, g)_h = h \sum_{k=0}^{N-1} \sum_{j=1}^{J} w_j (f\bar{g})(kh + \xi_j h), \tag{2.1}$$

where $0 < \xi_1 < \xi_2 < \cdots < \xi_J < 1$, $w_j > 0$, and $\sum_{j=1}^{J} w_j = 1$. The inner product integral is therefore approximated by the composite rule that results from applying to each sub-interval $[kh, (k+1)h]$ a suitably scaled version of a specially designed J-point rule

$$Qz = \sum_{j=1}^{J} w_j z(\xi_j). \tag{2.2}$$

Let us write (1.9) symbolically as

$$Kz = f. \tag{2.3}$$

Then the method of Sloan and Burn [14] takes the form: with an appropriate choice of Q (see below), find $z_h \in T_h$ such that

$$(K_h z_h, \chi)_h = (f, \chi)_h \quad \forall \chi \in S_h^r, \tag{2.4}$$

where

$$(K_h z)(x) = h \sum_{k=0}^{N-1} \log |\alpha(x) - \alpha(kh)| \, z(kh). \tag{2.5}$$

Theoretical results for the method of Sloan and Burn were previously known only for smooth curves and low orders of convergence. The known results include the following.

Theorem 2.1 *[14, 13] Suppose that Γ is a smooth curve, and that $\alpha \in C^\infty$ and $|\alpha'(x)| \neq 0$. Assume also that r is even, and that $J \geq 1$, with the case $J = 1$ and $\xi_1 = 1/2$ excluded. Define*

$$p = \begin{cases} 3 & \text{if } J = 2,\ \xi_1 = 1/6,\ \xi_2 = 5/6, \\ 2 & \text{if } J = 1,\ \xi_1 = 1/6 \text{ or } 5/6, \\ 1 & \text{otherwise.} \end{cases} \tag{2.6}$$

Then for

$$s \geq -1,\ t > -1/2 \text{ and } s \leq t \leq s + p \tag{2.7}$$

there exists $C > 0$ such that (2.4) has a unique solution $z_h \in T_h$ for h sufficiently small, satisfying

$$\|z - z_h\|_s \leq C\, h^{t-s} \|z\|_t, \text{ for } z \in H^t. \tag{2.8}$$

Here H^s, for $s \in \mathbb{R}$, is the Sobolev space of 1-periodic functions (or distributions) with finite values of the norm

$$\|f\|_s^2 = |\hat{f}(0)|^2 + \sum_{m \in Z^*} |m|^{2s} |\hat{f}(m)|^2, \tag{2.9}$$

where

$$\hat{f}(m) = \int_0^1 f(x) e^{-2\pi i m x}\, dx. \tag{2.10}$$

The proof of Theorem 2.1 and of later theorems in this paper (and even the very definition of the modification of Jeon and Kimn) rests on the decomposition of K into a principal part A and a remainder B,

$$\begin{aligned} K &= A + B \\ &= A(I + M), \end{aligned} \tag{2.11}$$

where

$$(Az)(x) = -2 \int_0^1 \log |2e^{-1/2} \sin(\pi(x - y))| \, z(y)\, dy, \tag{2.12}$$

$$(Bz)(x) = -2 \int_0^1 \log \left| \frac{\alpha(x) - \alpha(y)}{2e^{-1/2} \sin(\pi(x - y))} \right| z(y)\, dy = \int_0^1 b(x, y) z(y)\, dy, \tag{2.13}$$

and $M = A^{-1}B$. It is well known that A has a simple effect on the trigonometric monomial $\phi_m = e^{2\pi imx}$, namely

$$A\phi_m = \frac{1}{\max(1,|m|)}\phi_m, \tag{2.14}$$

from which it follows that A is an isometry operator from H^s to H^{s+1} for $s \in \mathbb{R}$.

Corresponding to the decomposition (2.11) we may write

$$K_h = A_h + B_h \tag{2.15}$$

where from (2.5)

$$\begin{aligned} (A_h z)(x) &= -2h\sum_{k=0}^{N-1} \log\left|2e^{-1/2}\sin(\pi(x-kh))\right| z(kh), & (2.16)\\ (B_h z)(x) &= -2h\sum_{k=0}^{N-1} \log\left|\frac{\alpha(x)-\alpha(kh)}{2e^{-1/2}\sin(\pi(x-kh))}\right| z(kh) & (2.17)\\ &= h\sum_{k=0}^{N-1} b(x,kh)z(kh). \end{aligned}$$

The method of Sloan and Burn may now be written as: find $z_h \in T_h$ such that

$$((A_h+B_h)z_h,\chi)_h = ((A+B)z,\chi)_h \quad \forall\chi \in S_h^r. \tag{2.18}$$

To allow us to write the defining equation (2.18) in operator form, we define, as in [13], [10], an operator P_h with image T_h, where $P_h : H^s \to T_h$ for $s > 1/2$, is defined by

$$(P_h z,\chi) = (z,\chi)_h \quad \forall\chi \in S_h^r, \tag{2.19}$$

with the exact inner product on the left but the approximate inner product on the right. The following lemma, embracing Lemmas 1 and 2 in the Appendix of [13], shows that P_h is well defined and has optimal convergence properties:

Lemma 2.1 *[13] The operator $P_h : H^s \to T_h$ for $s > 1/2$ is well defined by (2.19), and for $0 \le s \le t \le s+r$ and $t > 1/2$ there exists $C > 0$ such that*

$$\|P_h z - z\|_s \le C\,h^{t-s}\|z\|_t \text{ if } z \in H^t. \tag{2.20}$$

The operator P_h has a convenient representation, given in [10], in terms of the spline basis functions ψ_μ, with the latter defined as in [1] by

$$\psi_\mu = \sum_{m\equiv\mu (\mathrm{mod}\, N)} \left(\frac{\mu}{m}\right)^r \phi_m \text{ if } \mu \in \Lambda_h\backslash\{0\}, \tag{2.21}$$

and $\psi_0 = \phi_0 = 1$. Since $(\phi_\nu, \psi_\mu) = \delta_{\nu\mu}$ for $\nu, \mu \in \Lambda_h$, it is easily seen that

$$P_h z = \sum_{\mu \in \Lambda_h} (z, \psi_\mu)_h \phi_\mu. \tag{2.22}$$

Note that P_h is not a projection operator, since $P_h P_h \neq P_h$.

With the aid of the operator P_h, equation (2.18) may be written as

$$(P_h(A_h + B_h)z_h, \chi) = (P_h(A+B)z, \chi) \quad \forall \chi \in S_h^r, \tag{2.23}$$

or equivalently as

$$P_h(A_h + B_h)z_h = P_h(A+B)z, \tag{2.24}$$

where the last step follows from the following elementary result:

Lemma 2.2 *For $w_h \in T_h$, if $(w_h, \chi) = 0$ $\forall \chi \in S_h^r$ then $w_h = 0$.*

Proof: Write

$$w_h = \sum_{\nu \in \Lambda_h} a_\nu \phi_\nu,$$

and use $(\phi_\nu, \psi_\mu) = \delta_{\nu\mu}$ for $\nu, \mu \in \Lambda_h$ to show $a_\mu = 0$ for $\mu \in \Lambda_h$. □

A key role in the analysis in this paper is played by the special case of (2.24) with $B = 0$, in which case $z_h \in T_h$ satisfies

$$P_h A_h z_h = P_h A z.$$

Theorem 2.1 for the case $B = 0$ holds not just for h sufficiently small, but for all h. Thus this theorem establishes the existence and approximation properties of a solution operator for the case $B = 0$ defined by

$$R_h = (P_h A_h)^{-1} P_h A,$$

where the inverse is to be taken in the space T_h. In more detail, we have:

Theorem 2.2 *[14] Assume that r is even, and that $J \geq 1$, with the case $J = 1$ and $\xi_1 = 1/2$ excluded. Then $P_h A_h$ is bijective on T_h, and so*

$$R_h = (P_h A_h)^{-1} P_h A \tag{2.25}$$

is well defined. Let p be defined as in (2.6). Then for

$$s \geq -1,\ t > -1/2,\ s \leq t \leq s + p \tag{2.26}$$

there exists $C > 0$ such that

$$\|z - R_h z\|_s \leq C\, h^{t-s} \|z\|_t \text{ for } z \in H^t. \tag{2.27}$$

If $p \geq 2$ then $R_h z = z$ for all constant functions z.

Note that R_h is not a projection operator, since $R_h R_h \neq R_h$. With the aid of the operator R_h, equation (2.24) is equivalent to

$$z_h + R_h A^{-1} B_h z_h = R_h (I + A^{-1} B) z. \tag{2.28}$$

This is the form of the Sloan and Burn method in which we begin our analysis in Section 4.

3 A modified method

We begin by observing that A, the operator defined by (2.12) has, because of (2.14) with $m = 0$, the representation

$$Az(x) = -2 \int_0^1 \log |2e^{-1/2} \sin(x-y)| \, (z(y) - z(x)) \, dy + z(x),$$

in which the singularity in the integral has been weakened by the process of 'substraction of the singularity'. This motivates the definition of a new discrete approximation to replace A_h,

$$\begin{aligned} (A_h^M z)(x) &= -2h \sum_{k=0}^{N-1} \log |2e^{-1/2} \sin(\pi(x - kh))| (z(kh) - z(x)) + z(x) \\ &= (A_h z)(x) + e_h(x) z(x), \end{aligned} \tag{3.1}$$

where

$$e_h(x) = 1 + 2h \sum_{k=0}^{N-1} \log |2e^{-1/2} \sin(\pi(x - kh))|. \tag{3.2}$$

The modified method of [10] is: find $z_h \in T_h$ such that

$$(K_h^M z_h, \chi)_h = (f, \chi)_h \quad \forall \chi \in S_h^r, \tag{3.3}$$

where

$$K_h^M = A_h^M + B_h; \tag{3.4}$$

or equivalently, find $z_h \in T_h$ such that

$$P_h K_h^M z_h = P_h K z. \tag{3.5}$$

Analogously to Section 2, let $R_h^M : H^s \to T_h$ for $s > 1/2$ be the solution operator for the problem

$$z_h \in T_h, \quad P_h A_h^M z_h = P_h A z. \tag{3.6}$$

Thus

$$R_h^M = (P_h A_h^M)^{-1} P_h A, \tag{3.7}$$

where the inverse is again taken in the space T_h. The following theorem of Jeon and Kimn [10] establishes the existence and approximation properties of R_h^M:

Theorem 3.1 *[10] Assume that r is even, and that $J \geq 1$, with the case $J = 1$ and $\xi_1 = 1/2$ excluded. Then $P_h A_h^M$ is bijective on T_h, and so*

$$R_h^M = (P_h A_h^M)^{-1} P_h A$$

is well defined. Let

$$p = \begin{cases} 5 & \text{if } J = 2,\ \xi_1 = \xi,\ \xi_2 = 1 - \xi, \\ 2 & \text{otherwise}, \end{cases} \tag{3.8}$$

where $\xi = 0.2308296503\ldots$ is the smallest zero of

$$G(x) = 2 \sum_{l=1}^{\infty} \frac{1}{l^3} \cos(2\pi l x). \tag{3.9}$$

Then for

$$s \geq -1,\ t > -1/2,\ s \leq t \leq s + p$$

there exists $C > 0$ such that

$$\|z - R_h^M z\|_s \leq C\, h^{t-s} \|z\|_t \text{ for } z \in H^t. \tag{3.10}$$

Moreover, $R_h^M z = z$ for all constant functions z.

The last statement in the theorem follows from (3.1), (3.6) and (2.12).

4 Stability

In this section we study the stability of the qualocation methods (2.4) and (3.3).

From here on we will not distinguish between the solution operators R_h and R_h^M because there is no difference in our stability and convergence analysis except through the different values of the parameter p in (2.6) and (3.8). We therefore write R_h for both solution operators and A_h for both A_h and A_h^M. To simplify our analysis we assume that Γ has a single corner at s_0, $s_0 = \alpha(0) = \alpha(1)$. To be more precise we assume that Γ is infinitely smooth, with the exception of one corner point s_0, and that in a neighborhood of s_0, Γ consists of two straight lines intersecting with an interior angle $(1 - \chi)\pi$. Consider a parametrization $\alpha_0 : [0, 1] \to \Gamma$ which is C^∞ on $[0, 1]$ and satisfies

$$|\alpha_0(x) - s_0| = cx,\ x \in [0, \varepsilon],\ \ |\alpha_0(x) - s_0| = c(1 - x),\ x \in [1 - \varepsilon, 1], \tag{4.1}$$

for some $c > 0$ and sufficiently small $\varepsilon > 0$. Defining γ as in (1.6), we choose the mesh-grading parametrization $\alpha(x) := \alpha_0(\gamma(x))$, which satisfies because of (4.1)

$$|\alpha_0(x) - s_0| = |\alpha_0(1 - x) - s_0|,\ x \in [0, \varepsilon]. \tag{4.2}$$

(Note that $|\alpha(x)-s_0| = c\gamma(x)$, $|\alpha(1-x)-s_0| = c(1-\gamma(x))$ and that $\gamma(x) = 1-\gamma(1-x)$.) The extension of our analysis to polygonal boundaries with multiple corners can be carried out with minor extra effort. We shall also assume throughout that the order of convergence parameter p is at least 2, so that R_h reproduces the constant functions.

For $r > 0$ sufficiently small, we define the truncation $T_r v$ as the 1–periodic extension of

$$(T_r v)(x) = \begin{cases} 0, & x \in (0,r) \cup (1-r,1), \\ v(x), & x \in (r, 1-r). \end{cases} \tag{4.3}$$

Then T_{i^*h} is the truncation operator with $r = i^*h$. As usual, stability can only be proved if we admit the possibility of modifying the approximation near the corner. Thus instead of (2.24) we consider the modified approximation

$$(P_h A_h + P_h B_h T_{i^*h}) z_h = (P_h A + P_h B) z, \tag{4.4}$$

where

$$B_h T_{i^*h} z = h \sum_{k=i^*}^{N-i^*-1} b(x, kh) z(kh).$$

Here, i^* is a positive integer independent of h, which represents the number of subintervals cut off around corners. In fact i^* appears only for theoretical purpose, and in our numerical experiments we get the stability of our numerical system with $i^* = 0$.

Multiplying equation (4.4) by $(P_h A_h)^{-1}$ and using the solution operator R_h, we obtain

$$z_h + R_h M_h T_{i^*h} z_h = R_h z + R_h M z, \tag{4.5}$$

where $M_h = A^{-1} B_h$, which replaces (2.28). In this section, we will prove the stability of (4.5) in H^0, i.e., we prove

$$\|(I + R_h M_h T_{i^*h}) z_h\|_0 \geq C \|z_h\|_0, \quad z_h \in T_h \tag{4.6}$$

for some constant $C > 0$ independent of h, provided i^* is sufficiently large. Then (4.5) is uniquely solvable for z_h, and so therefore is (4.4).

For the proof of (4.6) we now recall from [4, 5] some analytical results on Equation (1.9) or (2.3) which are needed in the convergence analysis of the qualocation method. The first theorem was proved in [4], using a decomposition of M into a Mellin convolution operator local to the corner and a compact operator on H^0.

Theorem 4.1 *The operators $I + M : H^0 \to H^0$ and $K : H^0 \to H^1$ are continuously invertible, and we have the strong ellipticity estimate*

$$Re((I + M + T)v, v) \geq C \|v\|_0^2, \quad v \in H^0,$$

with some compact operator T on H^0.

The next result, also taken from [4], shows that the unique solution of (1.9) is smooth provided the given data g in (1.3) is smooth and the grading exponent is sufficiently large. Let $H^l(\Gamma)$, $l > 0$, denote the restriction of the usual Sobolev space $H^{l+1/2}(\mathbb{R}^2)$ to Γ.

Theorem 4.2 *Let $l \in I\!N$, $q > (l + 1/2)(1 + |\chi|)$, and suppose that $g = f \circ \alpha^{-1} \in H^{l+5/2}(\Gamma)$. Then the unique solution of (1.9) satisfies $z \in H^l$. Moreover, there exists $\delta < 1/2$ such that*

$$D^m z(x) = O(|x|^{l-m-\delta}) \quad \text{as } x \to 0, \text{ for } \quad m = 0, ..., l. \tag{4.7}$$

The following result from [5] describes the properties of the kernel function $b(x, y)$ of the operator B defined in (2.13).

Theorem 4.3 *On each compact subset of $\mathbb{R}\times\mathbb{R}\setminus(\mathbb{Z}\times\mathbb{Z})$, the derivatives $D^i_x D^m_y b(x, y)$ of order $i + m \le q$ are bounded and 1-periodic. Moreover, for $x, y \in [-1/2, 1/2] \setminus \{0\}$, we have the estimates*

$$|b(x, y)| \le C|\log(|x| + |y|)|,$$
$$|D^i_x D^m_y b(x, y)| \le C(|x| + |y|)^{-i-m}, \quad 1 \le i + m \le q.$$

Let us now return to the modified approximation (4.5).

Lemma 4.1 *For fixed $q \ge 2$ and each $\varepsilon > 0$ there exists $i^* \ge 1$ such that*

$$\|(I - R_h)MT_{i^*h}v\|_0 \le \varepsilon\|v\|_0, \quad v \in H^0, \tag{4.8}$$

and for all h sufficiently small

$$\|(I + R_h MT_{i^*h})v\|_0 \ge C\|v\|_0, \quad v \in T_h, \tag{4.9}$$

where C is independent of h and v.

Proof: First consider (4.8). Note that the operator M takes the form (cf. [4])

$$M = A^{-1}B = -HDB + JB,$$

with $Dv(x) = v'(x)$, $Jv(x) = \hat{v}(0)$ and H the (suitably normalized) Hilbert transform

$$Hv(x) = \frac{1}{2\pi} p.v. \int_0^1 \cot(\pi(x - y))v(y)\, dy,$$

which is bounded in L^2. This representation, together with the approximation property (2.27) or (3.10), and the fact that $I - R_h$ annihilates the constants, yields

$$\|(I - R_h)MT_{i^*h}v\|_0 \le Ch\|DMT_{i^*h}v\|_0 \le Ch\|D^2BT_{i^*h}v\|_0, \quad v \in H^0.$$

Hence (4.8) holds if we can show

$$\|D^2 BT_{i^*h} v\|_0 \leq \frac{C}{i^*h} \|v\|_0, \quad v \in H^0, \tag{4.10}$$

where C is independent of i^*, h and v. But (4.10) is shown in [7] by using the fact that from Theorem 4.3 $D^2BT_{i^*h}$ is bounded by $(i^*h)^{-1}$ times an integral operator with a Mellin convolution kernel $y/(x+y)^2$, which is a bounded operator in $L^2(0,\infty)$. (cf. Theorem 2.3 in [7]).

Next we consider (4.9). Since, by Theorem 4.1, $I + M$ is strongly elliptic and invertible on H^0, we obtain stability of the finite section operators $T_\tau(I+M)T_\tau$ as $\tau \to 0$, which implies the estimate (see [4, Theorem 6])

$$\|(I + MT_\tau)v\|_0 \geq C\|v\|_0, \qquad v \in H^0, \quad \tau \leq \tau_0. \tag{4.11}$$

Therefore the inequality (4.8) implies (4.9). □

For our analysis, the following standard estimate for the trapezoidal rule is needed. Here $J_h v$ denotes the trapezoidal rule approximation to $Jv = \hat{v}(0)$, with steplength h.

Lemma 4.2 *Let $l \in \mathbb{N}$, and suppose that v has 1-periodic continuous derivatives of all orders $< l$ on $\mathbb{R}$ and that $D^l v$ is integrable on $(0,1)$. Then for h sufficiently small*

$$|J(v) - J_h(v)| \leq Ch^l \int_0^1 |D^l v(y)|\, dy,$$

where c does not depend on v and h.

The proof of Lemma 4.2 is based on the representation

$$J(v) - J_h(v) = h^l \int_0^1 P_l(y/h) D^l v(y)\, dy,$$

where P_l is some 1-periodic piecewise polynomial of degree l, see [3, Chap. 2.9].

The following lemma is the key to the stability of (4.5).

Lemma 4.3 *a)For fixed $q \geq 2$ and $i^* \geq 1$, and for all h sufficiently small,*

$$\|(M - M_h)T_{i^*h}u\|_0 \leq \frac{C}{i^*}\|u\|_0 + Ch\|Du\|_0, \quad u \in H^1. \tag{4.12}$$

b) For fixed $q \geq 2$ and each $\varepsilon > 0$ there exists $i^* \geq 1$ such that for all h sufficiently small

$$\begin{aligned} \|(M - M_h)T_{i^*h}M_hT_{i^*h}v\|_0 &\leq \varepsilon\|v\|_0, && (4.13)\\ \|(M - M_h)T_{i^*h}(I - R_h)M_hT_{i^*h}v\|_0 &\leq \varepsilon\|v\|_0. && (4.14)\end{aligned}$$

Proof: Using

$$A^{-1} = -HD + J$$

and the definition of $M_h = A^{-1}B_h$, we get

$$\|(M - M_h)T_{i^*h}u\|_0 \leq C\{\|D(B - B_h)T_{i^*h}u\|_0 + \|(B - B_h)T_{i^*h}u\|_0\}. \tag{4.15}$$

Furthermore using Lemma 4.2 (for $l = 1$ and the interval $(-1/2, 1/2)$) and Theorem 4.3 we can show, as in [7], that (4.15) implies (4.12). There it is shown that

$$\begin{aligned} &|D(B - B_h)T_{i^*h}u(x)| + |(B - B_h)T_{i^*h}u(x)| \leq \\ &\leq \; (C/i^*)\int_{J_{i^*h}} \frac{|y|}{(|x| + |y|)^2}|u(y)|dy + Ch\int_{J_{i^*h}} \frac{|u'(y)|}{|s| + |y|}dy, \quad x \in (-1/2, 1/2), \end{aligned} \tag{4.16}$$

where $J_{i^*h} = (-1/2, -i^*h) \cup (i^*h, 1/2)$. Hence taking L^2 norms and using the fact that an integral operator with Mellin convolution kernel $y^m/(x + y)^{m+1}$, $m \geq 0$, is bounded in $L^2(0, \infty)$ we get (4.12) from (4.16). To prove (4.13) we set $u = M_hT_{i^*h}v$ in (4.12). It is shown in [7] that with some constant C, independent of h and i^*,

$$\|M_hT_{i^*h}v\|_0 \leq C\|v\|_0, \tag{4.17}$$

and

$$\|DM_hT_{i^*h}v\|_0 \leq \frac{C}{i^*h}\|v\|_0. \tag{4.18}$$

With these two inequalities (4.13) follows for i^* sufficiently large. To prove (4.14) we set $u = (I - R_h)M_hT_{i^*h}v$ in (4.12) to obtain

$$\|(M - M_h)T_{i^*h}(I - R_h)M_hT_{i^*h}v\|_0 \leq \frac{C}{i^*}\|(I - R_h)M_hT_{i^*h}v\|_0 + Ch\|D(I - R_h)M_hT_{i^*h}v\|_0. \tag{4.19}$$

Using the approximation property (2.27) or (3.10) together with (4.18), the last expression can further be bounded by

$$\frac{Ch}{i^*}\|DM_hT_{i^*h}v\|_0 + Ch\|DM_hT_{i^*h}v\|_0 \leq Ch\|DM_hT_{i^*h}v\|_0 \leq \frac{C}{i^*}\|v\|_0, \tag{4.20}$$

which gives (4.14) for i^* sufficiently large. □

We are now in the position to prove stability of the fully discrete method (4.5).

Theorem 4.4 *Assume $q \geq 2$ and suppose that i^* is sufficiently large. Then the estimate*

$$\|(I + R_hM_hT_{i^*h})v\|_0 \geq C\|v\|_0, \qquad v \in T_h \tag{4.21}$$

holds for all h sufficiently small, where C is independent of v and h.

Proof: By (4.9) the operators

$$(I + R_hMT_{i^*h})^{-1} : T_h \to T_h, \qquad h \leq h_0$$

exist and are uniformly bounded with respect to the H^0 operator norm if i^* is large enough. Consider

$$C_h := I - (I + R_h M T_{i^*h})^{-1} R_h M_h T_{i^*h}.$$

We see that

$$C_h(I + R_h M_h T_{i^*h}) = I - D_h, \tag{4.22}$$

with

$$D_h := (I + R_h M T_{i^*h})^{-1} R_h (M_h - M) T_{i^*h} R_h M_h T_{i^*h}.$$

From (4.12) we have

$$\|(M_h - M) T_{i^*h} v\|_0 \leq C \|v\|_0, \quad v \in T_h. \tag{4.23}$$

This together with the uniform boundedness of R_h on H^0 gives that $R_h M_h T_{i^*h}$ and hence C_h are uniformly bounded, too. Furthermore from the expression above and (4.13) and (4.14), we have, for given $\varepsilon \in (0,1)$,

$$\begin{aligned}
\|D_h v\|_0 &\leq C\|R_h(M_h - M)T_{i^*h} R_h M_h T_{i^*h} v\|_0 \\
&\leq C\{\|(M - M_h)T_{i^*h} M_h T_{i^*h} v\|_0 + \|(M - M_h)T_{i^*h}(I - R_h) M_h T_{i^*h} v\|_0\} \\
&\leq \varepsilon \|v\|_0, \qquad v \in T_h, \quad h \leq h_0,
\end{aligned}$$

provided that i^* is sufficiently large. Hence $(I - D_h)^{-1}$ exists and is uniformly bounded for i^* sufficiently large, so that (4.22) yields the assertion of the theorem. $\square$

5 Convergence Analysis

Theorem 5.1 *Let $p \geq 2$ be the parameter given by (2.6) or (3.8), and let $l \in I\!N$ and $q > (l + 1/2)(1 + |\chi|)$. Suppose also that $g \in H^{l+5/2}(\Gamma)$ and that $i^* \geq 0$ is such that the stability property (4.21) holds. Then (4.5) has a unique solution for all h sufficiently small, and for $l \leq p$*

$$\|z - z_h\|_0 \leq C h^l, \tag{5.1}$$

where z denotes the solution of (1.9) and the constant C is independent of h.

Proof: Step 1. First we verify (5.1) if the stability property (4.21) holds for some $i^* \geq 1$. Note that Theorem 4.4 implies that property provided i^* is sufficiently large. We have

$$\|z - z_h\| \leq \|(I - R_h)z\|_0 + \|z_h - R_h z\|_0,$$

in which the first term is of order h^l by (2.27) or (3.10). Using (4.21), together with Equation (4.5) and the uniform boundedness of R_h, we obtain for the second term

$$\begin{aligned}
\|z_h - R_h z\|_0 &\leq C\|(I + R_h M_h T_{i^*h})(z_h - R_h z)\|_0 \\
&= C\|R_h(I + M)z - (I + R_h M_h T_{i^*h}) R_h z\|_0 \\
&\leq C\|Mz - M_h T_{i^*h} R_h z\|_0 \\
&\leq C\|(M - M_h T_{i^*h})z\|_0 + C\|M_h T_{i^*h}(z - R_h z)\|_0.
\end{aligned} \tag{5.2}$$

Again using (4.12) and the approximation property (2.27) or (3.10) we can estimate

$$\begin{aligned}\|M_hT_{i^*h}(z-R_hz)\|_0 &\leq \|MT_{i^*h}(I-R_h)z\|_0+\|(M-M_h)T_{i^*h}(I-R_h)z\|_0 \\ &\leq C\|(I-R_h)z\|_0+Ch\|D(I-R_h)z\|_0\leq Ch^l, \end{aligned} \tag{5.3}$$

since $z\in H^l$ (see Theorem 4.2).

The first term in (5.2) can be estimated as in [7] by using again Mellin technique arguments. There it is shown that $\|(M-M_hT_{i^*h})z\|_0\leq Ch^l$ since

$$|x|^{-m}D^{l-m}z\in H^0, \quad m=0,\dots,l, \tag{5.4}$$

due to (4.7).

Step 2. Now we prove (5.1) under the assumption that the stability property (4.21) holds for $i^*=0$, i.e., the unmodified approximation $I+R_hM_h$ is stable. In order to do so, we modify the operator B_h defined in (2.17) without changing the method (2.18) or (3.3). Let $0<\eta\leq\min(\xi_1,1-\xi_J)$, and set

$$(\tilde{B}_hz)(x)=(B_hz)(x),\ x\in[\eta h,1-\eta h],$$

and

$$\begin{aligned}(\tilde{B}_hz)(x)=h\sum_{k=1}^{N-1}b(x,kh)z(kh)+hb(\eta h,0)z(0), \\ x\in[0,\eta h]\cup[1-\eta h,1].\end{aligned}$$

Since $b(\eta h,0)=b(1-\eta h,0)$ by (4.2) and the definition of b in (2.13), we see that $\tilde{B}_hz$ is a 1–periodic continuous function, which coincides with B_hz at every point of the quadrature rule in (2.1). Further we observe from (2.24) and the definition (2.22) of P_h that

$$P_h(A_h+B_h)z_h=P_h(A_h+\tilde{B}_h)z_h=P_h(A+B)z,$$

hence (2.28) (or equivalently (4.5) for $i^*=0$) can be written

$$(I+R_h\tilde{M}_h)z_h=R_h(I+M)z,\ \text{with}\ \tilde{M}_h=A^{-1}\tilde{B}_h. \tag{5.5}$$

Thus by assumption the operators $I+R_h\tilde{M}_h$ are stable in H^0, and as in (5.2) we obtain the estimate

$$\|z_h-R_hz\|_0\leq C\|(M-\tilde{M}_h)z\|_0+\|\tilde{M}_h(z-R_hz)\|_0. \tag{5.6}$$

Note that $(M-\tilde{M}_h)z=(M-M_h)z$ because of $z(0)=0$, and as in the proof of Theorem 3.4 in [7] we can show that $\|(M-M_h)z\|_0$ is of order h^l, using (5.4). The last term in (5.6) is bounded by

$$C\|M(z-R_hz)\|_0+C\|(M-\tilde{M}_h)(z-R_hz)\|_0, \tag{5.7}$$

where the first term is again of order h^l. To complete the proof of (5.1), we have to show that the last term in (5.7) is of order h^l. The proof of this relies on the following analogue of the inequality (4.12): namely that for fixed $q \geq 2$ and all h sufficiently small

$$\|(M - \tilde{M}_h)u\|_0 \leq C\|u\|_0 + Ch\|u\|_1, \ u \in H^1. \tag{5.8}$$

This has the desired effect, since using (5.8) we now obtain (cf. (5.3))

$$\|(M - \tilde{M}_h)(z - R_h z)\|_0 \leq C\|z - R_h z\|_0 + Ch\|z - R_h z\|_1 \leq Ch^l.$$

It remains to prove the estimate (5.8). Applying (4.12) for $i^* = 1$, we get

$$\|(M - \tilde{M}_h)T_h u\|_0 = \|(M - M_h)T_h u\|_0 \leq C\|u\|_0 + Ch\|u\|_1. \tag{5.9}$$

So we are left with proving an analogous bound for the term $\|(M - \tilde{M}_h)(I - T_h)u\|_0$. Since $\tilde{M}_h = (-HD + J)\tilde{B}_h$ (cf. the proof of Lemma 4.1), we have

$$\begin{aligned} \|(M - \tilde{M}_h)(I - T_h)u\|_0 \leq \|\tilde{B}_h(I - T_h)u\|_0 \quad &+ \quad \|D(B - \tilde{B}_h)(I - T_h)u\|_0 \\ &+ \quad \|B(I - T_h)u\|_0. \end{aligned} \tag{5.10}$$

Furthermore, by the definition of $\tilde{B}_h$ and the kernel estimate of Theorem 4.3,

$$|\tilde{B}_h(I - T_h)u(x)| \leq Ch|\log|\eta h||\,|u(0)|, \ x \in (-\eta h, \eta h)$$

and

$$|\tilde{B}_h(I - T_h)u(x)| \leq Ch|\log|x||\,|u(0)|, \ x \in J_{\eta h} = (-1/2, 1/2)\backslash(-\eta h, \eta h),$$

which gives

$$\|\tilde{B}_h(I - T_h)u\|_0 \leq Ch|u(0)| \leq Ch\|u\|_1. \tag{5.11}$$

Moreover, using Theorem 4.3 and Lemma 4.2 we obtain

$$\begin{aligned} |D(B - \tilde{B}_h)(I - T_h)u(x)| \quad &= \quad |DB(I - T_h)u(x)| \\ &\leq \quad C \int_{-1/2}^{1/2} \frac{1}{(|x| + |y|)}|u(y)|\,dy, \ x \in (-\eta h, \eta h), \end{aligned}$$

and for $x \in J_{\eta h}$,

$$\begin{aligned} |D(B - \tilde{B}_h)(I - T_h)u(x)| \quad &\leq \quad Ch\Big\{ \int_{-h}^{h} \frac{1}{(|x| + |y|)^2}|u(y)|\,dy + \int_{-h}^{h} \frac{1}{(|x| + |y|)}|u'(y)|\,dy \Big\} \\ &\leq \quad C \int_{-h}^{h} \frac{1}{(|x| + |y|)}|u(y)|\,dy + Ch \int_{-h}^{h} \frac{1}{(|x| + |y|)}|u'(y)|\,dy. \end{aligned}$$

Taking L^2 norms we get as in the proof of Lemma 4.3

$$\|D(B - \tilde{B}_h)(I - T_h)u\|_0 \leq C\|u\|_0 + Ch\|u\|_1. \tag{5.12}$$

Combining (5.10)–(5.12) with the fact that B is bounded on H^0 finally gives

$$\|(M - \tilde{M}_h)(I - T_h)u\|_0 \le C\|u\|_0 + Ch\|u\|_1,$$

which finishes the proof of (5.8). □

In many applications integral functionals of z are required. This happens, for example, when the solutions of boundary value problems are represented by interior potentials. These potentials may be written as smooth linear functionals (z, v) of the solution z of (1.9), if v is sufficiently smooth. For the purpose of studying such linear functionals we assume that (4.5) with $i^* = 0$ is stable in H^0 so that for given $f \in H^1$ a unique solution $z_h \in T_h$ of (2.4) exists for all h sufficiently small.

Theorem 5.2 *Suppose that Theorem 5.1 holds with $i^* = 0$ and that $v \circ \alpha^{-1} \in H^{l+5/2}(\Gamma)$. Then, for $l \le p - 1$ we have the error estimate*

$$|(z - z_h, v)| = O(h^{l+1}) \text{ as } h \to 0. \tag{5.13}$$

Remark. Theorems 5.1 and 5.2 hold, in particular, for the discrete trigonometric collocation method considered in [7], without any restriction on l.

Proof of Theorem 5.2: Let w be the unique solution of $Kw = v$ with $v \in H^{l+5/2}(\Gamma)$. Then Theorem 4.2 implies $w(s) = O(|s|^{l-\delta})$ as $s \to 0$ for some $\delta < 1/2$. Furthermore, since $K = A(I + M)$ and since A and K are self-adjoint we obtain

$$\begin{aligned} (z - z_h, v) &= ((I + M)(z - z_h), Aw) \qquad (5.14)\\ &= ((I - R_h)(I + M)(z - z_h), Aw) + (R_h(I + M)(z - z_h), Aw). \end{aligned}$$

From (5.5) one derives

$$R_h(I + M)(z - z_h) = (z_h - R_h z_h) + (R_h \tilde{M}_h z_h - R_h M z_h).$$

Therefore we get

$$\begin{aligned} (z - z_h, v) &= ((I - R_h)(I + M)(z - z_h), Aw) + (z_h - R_h z_h, Aw)\\ &\quad + (R_h(\tilde{M}_h z_h - M z_h), Aw)\\ &= P_1 + P_2 + P_3. \end{aligned}$$

Since R_h is bounded on H^0, its conjugate R_h^* exists and is bounded on H^0. Then with Theorem 4.1 and the approximation property of R_h in the H^{-1} norm we have

$$\begin{aligned} |P_1| = |((I - R_h)(I + M)(z - z_h), Aw)| &= |((I + M)(z - z_h), (I - R_h^*)Aw)|\\ &\le C\|z - z_h\|_0 Ch\|Aw\|_1 \qquad (5.15)\\ &\le Ch\|z - z_h\|_0\|w\|_0\\ &\le C\, h^{l+1}. \end{aligned}$$

Using $z_h - R_h z_h = (z - R_h z) + (I - R_h)(z_h - z)$, we have

$$\begin{aligned} |P_2| = |((z_h - R_h z_h), Aw)| &\le |((z - R_h z), Aw)| + |((I - R_h)(z_h - z), Aw)| \\ &\le \|z - R_h z\|_{-1} \|w\|_0 + Ch\|z_h - z\|_0 \, \|w\|_0 \qquad (5.16) \\ &\le Ch^{l+1}. \end{aligned}$$

For P_3 we have

$$\begin{aligned} |P_3| &\le |((\tilde{M}_h z_h - M z_h), Aw)| + |((I - R_h)(\tilde{M}_h z_h - M z_h), Aw)| \\ &= Q_1 + Q_2, \end{aligned}$$

where

$$\begin{aligned} Q_1 = |((\tilde{M}_h - M) z_h, Aw)| &= |(A^{-1}(\tilde{B}_h - B) z_h, Aw)| \\ &= |((\tilde{B}_h - B) z_h, w)| \\ &\le |((B_h - B) z_h, w)| + |((B_h - \tilde{B}_h) z_h, w)|. \qquad (5.17) \end{aligned}$$

Now by Lemma A.1 in the Appendix we have, for $\nu > 1/2$,

$$\begin{aligned} |(B_h - B) z_h(x)| &= \left| \sum_{j=0}^{N-1} b(x, jh) z_h(jh) - \int_0^1 b(x,y) z_h(y)\, dy \right| \\ &\le Ch^{\nu} \|b_x\|_{\nu} \|z_h\|_0, \end{aligned}$$

where $b_x(y) = b(x,y)$. Taking $\nu = l+1$, we find

$$\begin{aligned} \|b_x\|_{l+1}^2 &= |\hat{b}_x(0)|^2 + C\|D^{l+1} b_x\|_0^2 \le C + C \int_{-1/2}^{1/2} \frac{1}{(|x| + |y|)^{2l+2}}\, dy \\ &\le \frac{C}{|x|^{2l+1}}, \; x \in (-1/2, 1/2) \backslash \{0\}. \end{aligned}$$

Thus

$$|((B_h - B) z_h(x)) w(x)| \le Ch^{l+1} \|z_h\|_0 \frac{|w(x)|}{|x|^{l+1/2}},$$

and therefore, because $w(x) = O(|x|^{l-\delta})$ with $\delta < 1/2$

$$\int_{-1/2}^{1/2} |((B_h - B) z_h(x)) w(x)|\, dx \le Ch^{l+1} \|z_h\|_0 \le Ch^{l+1} \|z\|_0, \qquad (5.18)$$

where we have used the stability of (5.5). Furthermore, since

$$\begin{aligned} |(B_h - \tilde{B}_h) z_h(x)| &\le h(|b(x,0)| + |b(\eta h, 0)|)\, |z_h(0)| \\ &\le Ch |\log|x||\, |z_h(0)|, \; x \in (-\eta h, \eta h), \end{aligned}$$

and

$$B_h z_h(x) = \tilde{B}_h z_h(x), \; x \in J_{\eta h},$$

we obtain for some $\delta < 1/2$

$$\begin{aligned} |((B_h - \tilde{B}_h)z_h, w)| &\leq Ch|z_h(0)| \int_{-\eta h}^{\eta h} |y|^{l-\delta}\, dy \\ &\leq Ch^{l+2-\delta}|(z - z_h)(0)| \leq Ch^{l+1}\|z - z_h\|_{1-\delta}. \end{aligned} \tag{5.19}$$

Here we have used the fact that $z(0) = 0$ and Sobolev's embedding theorem. Applying the approximation property of R_h, the inverse property of T_h and (5.1), we can further estimate

$$\begin{aligned} \|z - z_h\|_{1-\delta} &\leq \|z - R_h z\|_{1-\delta} + \|z_h - R_h z\|_{1-\delta} \\ &\leq Ch^{\delta}\|z\|_1 + Ch^{\delta-1}\|z_h - R_h z\|_0 \leq Ch^{\delta}. \end{aligned} \tag{5.20}$$

Combining (5.18), (5.19) and (5.20) gives $Q_1 \leq Ch^{l+1}$. It remains only to prove an analogous result for Q_2. We have

$$\begin{aligned} Q_2 &\leq |((M - \tilde{M}_h)z, (I - R_h^*)Aw)| + |(M - \tilde{M}_h)(z - z_h), (I - R_h^*)Aw)| \\ &\leq Ch\|(M - \tilde{M}_h)z\|_0 + Ch\|(M - \tilde{M}_h)(z - z_h)\|_0, \end{aligned}$$

in which the first term is of order h^{l+1}; cf. the proof of Theorem 5.1. Finally, using the inequalities (5.8) and (5.1) and arguing as in (5.20), we see that the last term can be bounded by

$$\begin{aligned} Ch\|z - z_h\|_0 + Ch^2\|z - z_h\|_1 &\leq Ch^{l+1} + Ch^2\|z - z_h\|_1 \\ &\leq Ch^{l+1} + Ch^2\{\|z - R_h z\|_1 + Ch^{-1}\|z_h - R_h z\|_0\} \\ &\leq Ch^{l+1}. \end{aligned}$$

□

6 Numerical results

Consider a domain Ω with a re-entrant corner, enclosed by the curve

$$\Gamma : \quad (-(2/3)\sin((3/2)\tau), -\sin(\tau)), \quad 0 \leq \tau \leq 2\pi. \tag{6.1}$$

The angle of the re-entrant corner is $3\pi/2$. We also assume that the solution of (1.1) is

$$\phi(x_1, x_2) = \mathrm{Re}(\xi^{2/3}) = r^{2/3}\cos\frac{2\pi\theta}{3}, \quad \xi = x_1 + ix_2 = re^{i\theta}, \quad (x_1, x_2) \in \mathbb{R}^2\backslash\{0\}, \tag{6.2}$$

to give a realistic behaviour of ϕ at the corner. Because ϕ is the real part of an analytic function, it is clear that ϕ satisfies the Laplace equation in our domain. Let $\phi = g$ on Γ. Then g is smooth on Γ, and using the single layer potential to represent ϕ, we obtain the equation (1.3). The single layer density v will have regularity

$$v(x) = Cx^{-1/3} + \text{ a smoother function} \tag{6.3}$$

around the corner, where x represents the arc length from the corner. By a mesh grading transformation of order q, the new solution will be of the form,

$$z(x) = Cx^{(2q/3)-1} + \text{ a smoother function,} \tag{6.4}$$

which is much smoother than the original solution.

Let us define the functional

$$f^P(z) := -2\int_0^1 \log|P - \alpha(y)| z(y)\, dy, \qquad P \in \Omega, \tag{6.5}$$

and its approximation

$$f_h^P(z_h) := -\frac{2}{N+1}\sum_{k=0}^{N-1} \log|P - \alpha(kh)| z_h(kh). \tag{6.6}$$

Now the function $\log|P - \alpha(y)|$ is smooth if P is not a boundary point and Theorem 5.2 is applicable. In the following tables, the error e_N and the experimental convergence order β are defined as

$$e_N = |f^P(z) - f_h^P(z_h)|, \qquad P = (0.4, 0),$$

$$\beta = \frac{\log(e_{N_1}/e_{N_2})}{\log(N_1/N_2)}.$$

The experiments reported there show at least the order of convergence expected by the estimate (5.13). In our example with a re-entrant corner of $3\pi/2$ using a mesh grading with $q = 2, 3, 4, 5$ and a smooth right hand side we expect the solution z to belong to the Sobolev space H^l with $l = 5/6, 3/2, 13/6, 17/6$, respectively. Then Theorem 5.2 yields the theoretical convergence order $\beta_0 = l + 1 = 11/6, 5/2, 19/6, 23/6$, respectively. However, the numerical convergence rate observed in our experiments seems to be $O(h^{2l+1})$ instead of $O(h^{l+1})$.

Table 1 contains the numerical results for the discrete collocation discussed in [7]. Tables 2 and 3 show that we achieve the maximal orders of convergence 3 and 5 for the Sloan and Burn method and for the modified method, respectively.

If the grading parameter is $q = 3$, then the worst singularity $2q/3 - 1$ becomes smooth. In this case the modified method yields very fast convergence, whereas the other methods converge as expected. For $q = 2$ the convergence order of the Sloan and Burn method seems to exceed the maximal order 3 predicted by the theory.

Table 1: Discrete collocation

	$q=2$		$q=3$		$q=4$		$q=5$	
N	e_N	β	e_N	β	e_N	β	e_N	β
61	5.34 -6		2.61 -6		2.42 -6		2.34 -6	
		4.22		10.23		13.39		15.57
91	9.84 -7		4.36 -8		1.14 -8		7.97 -9	
		2.72		4.73		9.13		14.92
121	4.54 -7		1.13 -9		8.46 -10		1.13 -10	
		2.68		4.05		5.55		7.97
151	2.51 -7		4.61 -9		2.47 -10		1.94 -11	
		2.68		4.03		5.39		6.81
181	1.54 -7		2.22 -9		9.30 -11		5.64 -12	
		2.68		4.03		5.39		6.78
211	1.02 -7		1.20 -9		4.07 -11		1.99 -12	
		2.68		4.02		5.39		6.76
241	7.16 -8		7.01 -10		1.99 -11		8.12 -13	
		2.67		4.02		5.39		6.78
271	5.23 -8		4.38 -10		1.06 -11		3.67 -13	
		2.67		4.01		5.37		6.69
301	3.95 -8		2.87 -10		6.03 -11		1.82 -13	
		2.67		4.01		5.37		6.72
331	3.07 -9		1.96 -10		3.62 -12		9.60 -14	
		2.67		4.01		5.37		7.10
361	2.43 -9		1.38 -10		2.27 -12		5.18 -14	
β_0		1.83		2.50		3.17		3.83

Table 2: Method of Sloan and Burn of order 3

	$q=2$		$q=3$		$q=4$		$q=5$	
N	e_N	β	e_N	β	e_N	β	e_N	β
61	1.61 -5		1.73 -5		1.52 -5		1.28 -6	
		3.52		3.39		3.43		3.49
91	3.93 -6		4.48 -6		3.86 -6		3.16 -6	
		3.12		3.01		3.01		3.01
121	1.61 -6		1.89 -6		1.64 -7		1.34 -6	
		3.12		3.00		3.00		3.00
151	8.08 -7		9.76 -7		8.42 -7		6.90 -7	
		3.13		3.00		3.00		3.00
181	4.58 -7		5.67 -7		4.86 -7		4.01 -7	
		3.14		3.00		3.00		3.00
211	2.83 -7		3.58 -7		3.08 -7		2.53 -7	
		3.15		3.00		3.00		3.00
241	1.86 -7		2.40 -7		2.07 -7		1.70 -7	
		3.16		3.00		3.00		3.00
271	1.29 -7		1.69 -7		1.45 -7		1.19 -7	
		3.16		3.00		3.00		3.00
301	9.22 -8		1.23 -7		1.06 -7		8.71 -8	
		3.17		3.00		3.00		3.00
331	6.82 -8		9.27 -8		7.98 -8		6.55 -8	
		3.18		3.00		3.00		3.00
361	5.14 -8		7.15 -8		-8		5.05 -8	
β_0		1.83		2.50		3.00		3.00

Table 3: Modified method of order 5

	$q=2$		$q=3$		$q=4$		$q=5$	
N	e_N	β	e_N	β	e_N	β	e_N	β
61	1.90 -6		2.58 -6		2.47 -6		2.31 -6	
		5.16		11.57		12.07		15.57
91	2.41 -7		2.52 -8		1.98 -8		4.55 -9	
		2.34		6.91		6.65		6.93
121	1.24 -7		3.51 -9		2.97 -9		6.31 -10	
		2.56		6.05		5.03		4.88
151	7.02 -8		9.20 -10		9.75 -10		2.14 -10	
		2.60		6.60		4.99		5.04
181	4.37 -8		2.78 -10		3.95 -10		8.58 -10	
		2.63		7.66		4.99		5.03
211	2.92 -8		8.60 -11		1.84 -10		3.96 -11	
		2.64		10.26		4.99		5.03
241	2.06 -8		2.20 -11		9.46 -11		2.03 -11	
		2.65		43.63		4.99		5.03
271	1.51 -8		1.32 -13		5.27 -11		1.12 -11	
		2.65				4.99		5.04
301	1.14 -8		7.39 -12		3.12 -11		6.62 -12	
		2.66				4.99		5.05
331	8.87 -9		9.13 -12		1.94 -11		4.10 -12	
		2.66				4.99		5.05
361	7.04 -9		8.86 -12		1.26 -11		2.65 -12	
β_0		1.83		2.50		3.17		3.83

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Appendix

Lemma A.1 *Assume $\nu > 1/2$ and $\phi \in H^\nu$. Then*

$$\left| \int_0^1 \phi(t)u(t)\, dt - h \sum_{j=0}^{N-1} (\phi u)(jh) \right| \le C\, h^\nu \|\phi\|_\nu \|u\|_0$$

holds for all $u \in T_h$, with C independent of u.

Remark. This is an improved version of Lemma 4 in the Appendix of Saranen and Sloan [13].

Proof: As in Saranen and Sloan, we observe that

$$\int_0^1 \phi(t)u(t)\, dt = \sum_{n\in\Lambda_h} \hat{u}(n)\hat{\phi}(-n),$$

in which the right-hand side is a finite sum because $u \in T_h$. On the other hand

$$h \sum_{j=0}^{N-1} (\phi u)(jh) = \sum_{k\in\mathbb{Z}} (\widehat{\phi u})(kN),$$

where the latter series is absolutely convergent since $\phi u \in H^\nu$ with $\nu > 1/2$. Now we easily obtain, using again $u \in T_h$,

$$(\widehat{\phi u})(kN) = \sum_{n\in\Lambda_h} \hat{\phi}(kN-n)\hat{u}(n),$$

which gives

$$\sum_{k\in\mathbb{Z}} (\widehat{\phi u})(kN) = \sum_{n\in\Lambda_h} \hat{u}(n) \left[\hat{\phi}(-n) + \sum_{k\neq 0} \hat{\phi}(kN-n)\right].$$

Altogether there follows

$$\int_0^1 \phi(t)u(t)\,dt - h\sum_{j=0}^{N-1} (\phi u)(jh) = -\sum_{n\in\Lambda_h} \hat{u}(n) \sum_{\substack{m\equiv -n\\ m\neq -n}} \hat{\phi}(m),$$

so

$$\left|\int_0^1 \phi(t)u(t)\,dt - h\sum_{j=0}^{N-1} (\phi u)(jh)\right| \leq \left(\sum_{n\in\Lambda_h} |\hat{u}(n)|^2\right)^{1/2} \left(\sum_{n\in\Lambda_h}\left(\sum_{\substack{m\equiv -n\\ m\neq -n}} |\hat{\phi}(m)|\right)^2\right)^{1/2}.$$

But

$$\sum_{\substack{m\equiv -n\\ m\neq -n}} |\hat{\phi}(m)| = \sum_{\substack{m\equiv -n\\ m\neq -n}} |m|^{-\nu}|m|^{\nu}|\hat{\phi}(m)| \leq \underbrace{\left(\sum_{\substack{m\equiv -n\\ m\neq -n}} |m|^{-2\nu}\right)^{1/2}}_{C\,h^\nu,\text{ for }\nu>1/2} \left(\sum_{\substack{m\equiv -n\\ m\neq -n}} |m|^{2\nu}|\hat{\phi}(m)|^2\right)^{1/2},$$

so

$$\begin{aligned}\left|\int_0^1 \phi(t)u(t)\,dt - h\sum_{j=0}^{N-1} (\phi u)(jh)\right| &\leq \|u\|_0 \left(\sum_{n\in\Lambda_h} C\,h^{2\nu} \sum_{\substack{m\equiv -n\\ m\neq -n}} |m|^{2\nu}|\hat{\phi}(m)|^2\right)^{1/2} \\ &\leq C\,h^\nu\|\phi\|_\nu\|u\|_0.\end{aligned}$$

□